# Analysis, control and stabilization of evolution equations arising in real-world problems

Siena, April 14th-16th, 2025

**Analysis, control and stabilization of evolution equations arising in real-world problems**

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Department of Information Engineering and Mathematics - University of Siena

**Speakers:**

Mohammad Akil
Jochen Bröcker
Fabio Camilli
Marco Caponigro
Chiara Cicolani
Elisa Continelli
Emmanuelle Crepeau
Anna Doubova
Franco Flandoli
Giuseppe Floridia
Fausto Gozzi
Sarah Ismail
Ibtissam Issa
Tobias Kuna
Marta Leocata
Paola Loreti
Serge Nicaise
Patrick Martinez
Michele Palladino
Eugenio Pozzoli
Daniela Sforza
Judith Vancostenoble
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# Analysis, control and stabilization of evolution equations arising in real-world problems

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**ABSTRACTS - TALKS**

## The Influence of Active Boundary Control and Lower-Order Passive Distributed Dampers on the Stability of Serially Connected Elastic and Piezoelectric Smart Sensor Designs

*Mohammad Akil*
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In this talk, we analyze the stability of longitudinal vibrations in transmission problems for two smart-system designs:

(i) a serially connected elastic–piezoelectric–elastic system.

(ii) a serially connected piezoelectric–elastic–piezoelectric system.

For (i), we examine two scenarios: first, a local damping acting solely on the elastic part; second, an active boundary damping applied at the extremity of the right elastic part and at the nodes.

For (ii), we consider two cases: the first involves viscous damping on each piezoelectric beam, ensuring both strong and exponential stability; the second consists of various configurations combining viscous damping with boundary and/or node electrical damping.

In both (i) and (ii), we establish exponential and polynomial stability, with or without arithmetic conditions on the physical parameters. Finally, we validate our findings numerically.

## Data Assimilation

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"Data assimilation" is a term used in the geosciences to refer to reconstructing the current state of a dynamical system from current and past observations of that system. By "observation" we mean the value of some (typically not injective) function of the current state of the system. The system itself as well as the observations might be perturbed by noise. The underlying dynamics as well as the observation function might contain unknown parameters which might be of interest as well. In geophysical applications, the underlying dynamics are typically infinite dimensional, while observations are finite dimensional (albeit with very large dimensions). Data assimilation is of great practical interest for instance in operational weather forecasting.

In this contribution, I will try to give a brief overview over data assimilation from a mathematical perspective. There is no single mathematical paradigm that covers data assimilation entirely, but a wide range of different mathematical approaches to data assimilation exist. These fall broadly into deterministic and stochastic approaches, and I will discuss both of them. Several important results will be discussed and a number of open questions and problems presented.

**Stationary Mean Field Games on networks with sticky transition conditions**

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We investigate stochastic Mean Field Games on networks with sticky transition conditions, where agents' dynamics are governed by a diffusion process that can reside for finite time both within the interior of edges and at vertices. This framework imposes constraints on the second-order derivatives of the associated generator, resulting in an invariant measure that combines an absolutely continuous component along the edges with Dirac measures at the vertices. Furthermore, the value function, which solves the Hamilton-Jacobi-Bellman equation, satisfies generalized Kirchhoff conditions at the vertices, capturing the complex interplay of strategic behavior and diffusion across the network.

**Finite dimensional Galerkin approximations for bilinear quantum systems**

*Marco Caponigro*
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Several infinite dimensional bilinear quantum systems encountered in the physics literature can be described, with good precision, by appropriate finite dimensional approximations. We present a regularity condition sufficient for the existence of these approximations. We also show a counterexample of a system that is approximately controllable while its infinite dimensional dynamics cannot be precisely described by its finite dimensional Galerkin approximations.

Joint work with Nabile Boussaïd and Thomas Chambrion.

**Second-order alignment models with non-universal interaction, time delay and communication failures**

*Chiara Cicolani*
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In this talk, we present the second-order alignment models with non-universal interaction, time delay and possible lack of connection between the agents. We prove the unconditional flocking for solutions to such systems. This is done, as for the first-order model (see [1]), by assuming that the digraph that describes the interaction among the agents is strongly connected and using a Persistence Excitation Condition. With respect to the first-order model, a more careful analysis is needed in order to establish the exhibition of flocking and therefore the main differences with the first-order model will be emphasized. Some numerical simulations are presented to verify the theoretical results and to justify possible applications.

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**First-order alignment models with non-universal interaction, time delay and communication failures**

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We deal with first-order alignment models with non-universal interaction, time delay, and possible lack of connection. Namely, we investigate the situation in which the agents involved in an opinion formation process are not linked with all the other components of the system and also agents that generally able to exchange information among themselves can suspend their interaction at certain times. Due to the non-universal interaction, we consider a graph topology over the model structure. Under a so-called Persistence Excitation Condition, we prove that solutions to such models converge exponentially fast to consensus, provided that the digraph describing the interaction is strongly connected.

Joint work with C. Cicolani and C. Pignotti.

**Carleman-based reconstruction algorithm on a wave network**

*Emmanuelle Crepeau*
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In this talk, I will present the inverse problem of reconstructing a potential-type coefficient for a 1D wave equation set on a tree-like network. The internal nodes of this network satisfy Kirchoff-type transmission conditions, while the external nodes are subject to Dirichlet conditions. The question we explored, together with L. Baudouin, M. de Buhan, and J. Valein, is how to reconstruct the potential on each branch of the network by measuring only the flux at the external nodes. To address this, we propose a reconstruction algorithm based on Carleman-type inequalities, which also enables us to achieve Lipschitz stability for the inverse problem. I will discuss both the theoretical results obtained, including the global convergence of the algorithm, as well as numerical results.

Bibliography: Lucie Baudouin, Maya de Buhan, Emmanuelle Crépeau, Julie Valein. Carleman-Based Reconstruction Algorithm on a wave Network. 2023. ⟨hal-04361363⟩

**Identification of degeneracy in parabolic equations**

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We address an inverse problem involving the reconstruction of a degeneracy point in the diffusion coefficient of a one-dimensional parabolic equation using measurements of the normal derivative on one side of the boundary. Our study focuses on the sensitivity of this inverse problem to the initial data. We establish sufficient conditions on the initial data to ensure uniqueness and stability with a single-point measure, and provide examples of both positive and negative results. These theoretical conclusions are supported by numerical evidence. We also present more general uniqueness results for the identification of both degeneracy and initial data by boundary measurements distributed over time. The method of proof is based on the representation of the solution by means of Bessel functions of the first kind.

This is a joint work with P. Cannarsa and M. Yamamoto.

**Hasselmann stochastic theory and application to the variability of short-term climate**

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Klaus Hasselmann (Nobel Prize in Physics 2021 for his contribution to climate research), wrote a seminal paper in 1976, which originated a lot of research. We apply his ideas to an Energy Balance Model in order to show that the climate variability (meant for short-term climate, of the order of the averages over an year), may change in time, as a consequence of the more standard change in time of the mean values. The activities mentioned herein are performed in the framework of the project NOISYFLUID, Noise in Fluids (GA 101053472).

**Carleman Estimates for Transport Equations**

*Giuseppe Floridia*
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In this talk, we present some results on inverse problems for first-order hyperbolic equations using Carleman estimates. In particular, we introduce a Carleman estimate, which we apply to certain inverse problems related to the determination of an initial value and a spatial factor of a source term. As a result, we establish Lipschitz stability estimates for these inverse problems.

This is a joint work with Piermarco Cannarsa and Masahiro Yamamoto.

**Long-time behaviour of the solution for the quasi-geostrophic coupled atmosphere-ocean model**

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In this talk, we will show that the model presented in Vannitsem et al. in [1FF] for large times admits a global attractor, and that all its solutions can be uniformly bounded in time in some suitable functional spaces. Following the approach used by Temam in [2FF] for 2D Navier Stokes, we will also show that the attractor is finite dimensional, and we will use determining functionals in order to prove that the thermodynamics of the ocean is governed for large times by the dynamics of the rest of the system. In the last part of the talk we will also, using the regularity of the solution of the model, derive conditions under which the ocean and atmosphere's temperature (as absolute temperatures in Kelvin) stay positive under the time evolution.

This is a joint work with Tobias Kuna (Univaq) and Giulia Carigi (Indiana).

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[2] Roger Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, 1988.

**On some Mathematical Models for Biodiversity and Agroecology**

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In this talk we present some ideas on how to model the evolution of biodiversity in the Antropocene, (the era where the presence of homo sapiens is modifying deeply the biosphere) and its relations with the economic variables. In particular we focus on two aspects:

- on one side the relations of biodiversity loss, deforestation and agriculture in an optimal control model;
- on the other side the space-time evolution of biodiversity in a spatial mean field game.

Papers in progress with E. Augeraud, R. Boucekkine, A. Calvia, D. Ghilli, M. Leocata, F. Masiero

**Harnessing Control Theory for Climate: Null-Controllability for Non-Autonomous Degenerate Parabolic Equations**

*Sarah Ismail*
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Global warming - the gradual heating of Earth's surface, oceans and atmosphere - is one of the most concerning problems facing all living beings. Since 1979, the Arctic has warmed nearly four times faster than the rest of the planet.

In order to regulate the global temperature and recover the equilibrium of the "Earth's radiation budget", we need to balance the amount of solar radiation reflected by the amount absorbed by Earth's surface. To simulate the interaction of basic climate system parameters on Earth, climatologists Budyko and Sellers independently introduced a classical energy balance model in 1969, which can be rewritten as the following one-dimensional nonlinear degenerate parabolic equation

$$u_t(t,x) - \rho_0 \left(a(x)u_x\right)_x = R_a(t,x,u) - R_e(t,x,u),$$

where $u(t,x)$ is the surface temperature averaged over longitute, $a(x) = 1 - x^2$ ($x = \sin\varphi \in [-1,1]$, with $\varphi$ being the latitude) and $\rho_0$ is a positive parameter.

In this talk, I will be presenting new results on a non-autonomous degenerate parabolic equation in divergence or in non-divergence form with various boundary conditions. First, we study the null-controllability for the considered problem with Dirichlet or Neumann boundary conditions. Then, we consider the Robin boundary conditions being convenient to model the convection occuring at the surface of the planet. Thus, we provide new Carleman estimates for the associated non-autonomous adjoint system under these boundary conditions.

This talk is based on joint works with Genni Fragnelli (Università degli Studi di Siena) and Mohammad Akil (Université Polythechniques Hauts-de-France).

**Dynamic Stabilization of Two-String Systems with Dynamical Interior Mass: Unveiling the Role of Higher-Order Nodal Damping**

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This paper investigates the stabilization of a two-string system with a single dynamical interior mass, providing refined results and novel insights into the interplay of boundary damping, higher-order nodal damping (angular velocity), and lower-order nodal damping (tip velocity). The analysis reveals that exponential stability is unconditionally achieved when boundary damping is combined with higher-order nodal damping. In contrast, lower-order nodal damping, even with boundary damping, ensures only polynomial stability with a decay rate of $t^{-1}$. Notably, the case of boundary damping without nodal damping aligns with the pioneering work [1], confirming that polynomial stability is still obtained. However, our findings demonstrate a significant improvement over [2], where a slower decay rate of $t^{-\frac{1}{2}}$ was reported, whereas we achieve a decay rate of $t^{-1}$ under similar configurations. In the absence of boundary damping, the results show that higher-order nodal damping can independently achieve exponential stability under a resonance-avoiding condition, characterized by specific geometric and material parameter ratios. Conversely, lower-order nodal damping alone can at most ensure polynomial stability with a decay rate of $t^{-1}$. These findings highlight the superior stabilization effects of higher-order damping mechanisms and provide a structured framework for optimizing stability and energy dissipation in hybrid two-string systems with a dynamical interior mass.

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[2] W. Littman and S. W. Taylor. *Boundary feedback stabilization of a vibrating string with an interior point mass.* Springer, 2002.

**Well-posedness of quasi-geostrophic coupled atmosphere ocean model.**

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In order to understand the weather on longer times scales, like for example, on low frequency variability on annual and decadal scales, but also on climatic and paleoclimatic scales, it is inevitable to incorporate ocean atmosphere coupling and thermodynamical effects into the modelling. As a first step, we establish the well-posedness of a model consisting of a system of PDEs describing an atmosphere via two quasi-geostrophic layers coupled to a further quasi-geostrophic layer modelling a deep ocean. Furthermore, there are two transport reaction-diffusion PDEs describing the development of the atmosphere and ocean temperature. More specifically, we consider the model which Vannitsem et al. in [1K] used (not linearising the infrared radiation terms), which is based on a series of previous models, to mention a few Charney and Strauss '80, Reinhold and Pierrehumbert '82; Pierini '11. A lot of work has been done about this model by meteorologist using physical and numerical considerations. We consider these model from a mathematical analytical angle proving existence of weak, strong, and classical solution. Furthermore, we establish weak-uniqueness. An interesting aspect of the model is the asymmetry in the unknowns with respect to regularity, which leads to a not straight forward application of classical techniques for 2D-Navier-Stokes and reaction diffusion equations.

This is a joint work with Federico Fornasaro (La Sapienza) and Giulia Carigi (Indiana).

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[1] Lesley De Cruz, Jonathan Demaeyer, Stephane Vannitsem, The Modular Arbitrary-Order Ocean-Atmosphere Model: MAOOAM v1.0, Geoscientific Model Development, 2016.

### On some models of biodiversity evolution

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In the first part of the talk, we consider a N-players game where agents maximize a given utility function depending on local environmental quality (specific to each location) and on a global enviromental quality, which is the interaction term. We are interested in understanding whether taking into account global environmental quality (thinking globally) may better local environmental quality. We show that local environmental quality may not necessarily be better by this consideration. We validate our choices in the model by an empirical analysis. In the second part of the talk, we analyse the case in which there is an infinite number of locations (and then agents) and agents are still competitive, that is, the Mean Field Game associated. Finally, we study the case in which agents are cooperative and there is a social planner who optimizes, that is the Mean Field Control associated.

### Direct and inverse inequalities and boundary controllability of the wave equation

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In the talk I will give a brief overview of the multiplier method and the Fourier method to solve exact controllability for the wave equation with boundary controls. Then I will discuss some models arising in the real world, as viscoelastic materials with focus on the memory term and the related integro-differential equations with the integral term of convolution type (see [1], [2], [3]).

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[1] P. Loreti, D. Sforza, Hidden regularity for wave equations with memory, Riv. Mat. Univ. Parma, 7, No. 2 (2016), 391–405.

[2] P. Loreti, D. Sforza, Viscoelastic aspects of glass relaxation models. Phys. A, 526 (2019), 120768, 10 pp.

[3] P. Loreti, D. Sforza, Controllability for the Burgers model, Journal of Mathematical Analysis and Applications, Volume 531, Issue 2, Part 2, (2024), 127836.

### Parameter determination for some nonlinear parabolic equations

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We are interested in parameter determination for some nonlinear parabolic equations:

- the Fisher-KPP equation, modelling biological invasions,

- a 2-layer Energy Balance Climate Model, appearing in climate dynamics.

Some parameters (the growth rate for the Fisher-KPP model, the insolation function for the EBCM model) must be determined thanks to measurements of the solution, in order to study the long time behaviour of the solution. For the Fisher-KPP model, we obtain stability results ([4]) for these unknown parameters with respect to some suitable measurements, combining Carleman estimates with classical nonlinear tools (regularity results, maximum principles, Harnack inequalities). (This completes earlier reasults of [2,3].)

In the second part, we study the same question for a 2-layer EBCM (studied in [5,6] from a dynamical point of view), with tools developed in [1,6].

Joint works with Piermarco Cannarsa (*Univ. Roma 2*), Valerio Lucarini (Univ. Leicester), Cristina Urbani (Univ. Mercatorum) and Judith Vancostenoble (Univ. Toulouse 3).

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**Decay properties of the Maxwell system with conductivity**


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We discuss the linear autonomous Maxwell system with damping caused by a nonnegative conductivity $\sigma$. For the scalar wave equation it is well known that the location of the support of $\sigma$ often determines the resulting decay behavior. The Maxwell case is far less studied and poses additional difficulties. For instance, the charges (or divergence conditions) play a crucial role as they have to counteract the large kernel of the curl operator.

We present two recent results assuming that the permittivity $\varepsilon$ and permeability $\mu$ are constant. The first one concerns the strong stability in a quite general context. The main difficulty is the closedness of the range of the operator, which is obtained by proving as sort of Poincaré inequality. The second one concerns the polynomial decay if $\sigma$ is strictly positive on a strip of a cube. This fact follows from a resolvent estimate which is shown by means of the eigenfunctions of the undamped Maxwell problem.

This is a joint work with Roland Schnaubelt (Karlsruhe).

**Optimal Control and Reinforcement Learning**

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The talk discusses a framework to analyze certain model-based reinforcement learning algorithm. Roughly speaking, this approach consists in designing a model to deal with situations in which the system dynamics is not known and encodes the available information about the state dynamics that an agent has as a measure on the space of functions. In this framework, a natural question is if whether the optimal policies and the value functions converge, respectively, to an optimal policy and to the value function of the real, underlying optimal control problem as soon as more information on the environment is gathered by the agent. We provide a positive answer in the linear-quadratic case and discuss some results also in the control-affine nonlinear case.

**Small-time approximate controllability of bilinear Schrödinger and Liouville equations, and diffeomorphisms**

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In this talk we discuss bilinear Schrödinger and Liouville (Hamiltonian) equations. More precisely, we consider:

- **Schrödinger equations:**

$$i\partial_t \Psi(x,t) = \Big( -\Delta + V(x) + \sum_{j=1}^{m} u_j(t) W_j(x) \Big) \Psi(x,t), \tag{1}$$

  with wavefunction $\Psi(\cdot,t) \in L^2(M,\mathbb{C})$, and controls $u_j$, where $\Delta$ is the Laplace-Beltrami operator of a boundaryless Riemannian manifold $M$, and $V, W_1, \dots, W_j$ are smooth real-valued functions on $M$ (possibly unbounded). In analogy to classical mechanics, the generator $H = -\Delta + V(x) + \sum_{j=1}^{m} u_j(t) W_j(x)$ is often called the Hamiltonian.

- **Liouville equation:**

$$\partial_t \rho(q,p,t) = \vec{H}(q,p,u(t))\rho(q,p,t), \quad H = \frac{|p|^2}{2} + V(q) + \sum_{j=1}^{m} u_j(t) W_j(q), \tag{2}$$

  with density $\rho(t) \in L^p(T^*M), p \in [1,\infty)$, transported along an Hamiltonian vector field $\vec{H}$, with controls $u_j$, where the Hamiltonian function $H$ on the cotangent bundle $T^*M$ (with coordinates $(q,p)$) defines the Hamiltonian vector field as

$$\vec{H} := \{H, \cdot\} = p \cdot \nabla_q - \nabla_q V \cdot \nabla_p - \sum_{j=1}^{m} u_j(t) \nabla_q W_j \cdot \nabla_p,$$

  a first-order differential (transport) operator, and $\{\cdot,\cdot\}$ denotes the Poisson bracket.

We exhibit examples of Schrödinger and Liouville equations on $M = \mathbb{T}^d$ and $\mathbb{R}^d$ where we are able to prove global approximate controllability. The results hold in arbitrarily small times and do not require a discrete spectrum Hamiltonian. A key ingredient in the analysis is to link the controllability of the Schrödinger, resp. Liouville, equation with the controllability of the group of diffeomorphisms $\mathrm{Diff}_c^0(M)$ of $M$, resp. Hamiltonian diffeomorphisms $\mathrm{DHam}(T^*M)$ of $T^*M$.

The results on Schrödinger equations are based on two articles in collaboration with Karine Beauchard [1], [2].

The results on Liouville equations are based on ongoing projects in collaboration with Bettina Kazandjian and Mario Sigalotti.

## References


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**On time-fractional wave equations**


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To describe some viscoelastic materials, for which memory cannot be neglected, a power-law function is used in the constitutive assumptions. Associated with a power-law memory are usually integrable singular kernels, which naturally give rise to time-fractional differential equations, an expanding field of study with increasing relevance in both mathematics and its practical applications.

In this presentation, I will discuss recent findings (see [1], [2], [3]) on the well-posedness and regularity of solutions of the Caputo and Riemann-Liouville fractional equations. These results were obtained using the method of multipliers, combined with specialized techniques from interpolation theory.

## References


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**About some 2-layer energy balance climate model**


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We aim to study a two-layer energy balance model, that allows for vertical exchanges between a surface layer and the atmosphere. An essential parameter is the absorptivity of the atmosphere, denoted $\varepsilon_a$ whose value depends critically on greenhouse gases (increasing concentrations of $CO_2$ and $CH_4$ lead to a more opaque atmosphere with higher values of $\epsilon_a$).

We aim here to make a complete mathematical analysis of the model (before turning to inverse problems questions) by obtaining results of global existence vs blow-up (depending on the value of $\varepsilon_a$), describing the equilibrium points (recovering the multi-stability of climate) and their sensitivy with respect to $\varepsilon_a$ (observing the mathematical manifestation of the greenhouse effect for this toy model).

Joint works with: Piermarco Cannarsa (Univ. Roma 2), Valerio Lucarini (Univ. Leicester), Patrick Martinez (Univ. Toulouse), Cristina Urbani (Univ. Mercatorum).

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**Uniqueness of solutions to an elliptic and parabolic inequalities with rapid decays**


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First we consider an elliptic differential inequality:

$$|\Delta u(x)| \le C_0(|y|^{-\gamma}|u(x)| + |y|^{-\theta}|\nabla u(x)|)$$

in an exterior domain $\mathbb{R}^n \setminus \overline{U}$ with bounded domain $U$, where $x := (y, z) \in \mathbb{R}^n$ with $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{n-m}$ for given $m \in \{1, ..., n\}$, and $\gamma, \theta \in \mathbb{R}$ are constants. We prove that if $u(x)$ decays with some exponential rate in the $y$-coordinates and polynomial rate in the rest $z$-coordinates as $|x| \to \infty$, then $u \equiv 0$ in $\mathbb{R}^n \setminus \overline{U}$. The exponential and polynomial decay rates depend on $\gamma, \theta \in \mathbb{R}$ and $m \in \{1, 2, ..., n\}$. This is related to Landis' conjecture (e.g., [1], [2], [3]), and our result can restrict the exponential decay to some directions in $\mathbb{R}^n$. The key is a Carleman estimate (e.g., [4]) with typical cut-off arguments, and the method is applicable to various partial differential inequalities and equations.

Next we consider a parabolic differential inequality in a bounded smooth domain $\Omega$:

$$\begin{cases} \left|\partial_t u(x,t) - \sum_{i,j=1}^n \partial_i(a_{ij}(x,t)\partial_j u(x,t))\right| \le d_0(t)|\nabla u(x,t)| + d_1(t)|u(x,t)|, \\ \qquad\qquad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega\times(0,\infty)} = 0 \quad \text{or} \quad \partial_{\nu_A} u + qu = 0 \quad \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

with some function $d_0, d_1$. In general, if $u$ decays in $t$, then one can prove that $\|u(\cdot,t)\|_{L^2(\Omega)} = O(e^{-ct})$ with $c > 0$. By Carleman estimate, we prove that if $u$ decays faster, then $u = 0$ in $\Omega \times (0,\infty)$ with critical decay rates which are determined by the behavior of $d_0(t)$ and $d_1(t)$ as $t \to \infty$.

This is based on joint works with F. Gölgeleyen (Zonguldak Bülent Ecevit University, Türkiye) and O.Y. Imanuvilov (Colorado State University, USA).

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